Math 245C Lecture 4 Notes

Daniel Raban

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1 Minkowski's Inequality and The Marcinkiewicz Interpolation Theorem

1.1 Minkowski's inequality

Let $f : X \to \mathbb{C}$ be measurable. For $A > 0$, set $h_A = \phi_A \circ f$, $g_A = f - h_A$, where

$$\phi_A(z) = \begin{cases} z & |z| < A \\ \frac{z}{|z|}A & |z| \geq A. \end{cases}$$

Then

$$\lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & 0 < \alpha < A \\ 0 & \alpha > A, \end{cases}, \qquad \lambda_{g_A}(\alpha) = \lambda_f(A + \alpha).$$

Recall Minkowski's inequality:

Theorem 1.1 (Minkowski's inequality). *Let $1 \leq r < \infty$, and let $f : X \times Y \to [0, \infty]$. Then*

$$\int_Y \left(\int_X |f(x,y)|^r \, d\mu(x) \right)^{1/r} d\nu(y) \geq \left(\int_X \left(\int_Y f(x,y) \, d\nu(y) \right)^r d\mu(x) \right)^{1/r}.$$

1.2 The Marcinkiewicz interpolation theorem

Theorem 1.2 (Marcinkiewicz interpolation theorem). *Let $\mathfrak{F}$ be the set of measurable functions on Y. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ be real numbers such that $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$. Let $t \in (0,1)$, and let p, q be defined as*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Assume that $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to \mathcal{F}$ be sublinear and of weak type (q_0, p_0) and (q_1, p_1) (there are $c_0, c_1 > 0$ such that if $q_0, q_1 \neq \infty$, $(\alpha^{q_0} \lambda_{T(f)})^{1/q_0} \leq c_0 \|f\|_{p_0}$ and $(\alpha^{q_1} \lambda_{T(f)})^{1/q_1} \leq c_1 \|f\|_{p_1}$). Then the following hold:

1. *T is strong type (p,q) (there exists $B_p > 0$ such that $\|Tf\|_q \leq B_p \|f\|_p$ for all $f \in L^p(\mu)$).*

2. *If $p_0 < \infty$, then $\lim_{p \to p_0} B_p |p_0 - p| < \infty$. If $p_1 < \infty$, then $\lim_{p \to p_1} B_p |p_1 - p| < \infty$. If $p_0 = \infty$, (B_p) remains bounded as $p \to p_0$. If $p_1 = \infty$, (B_p) remains bounded as $p \to p_1$.*

Proof. We skip the proof in the case $p_1 = p_0$. Let us assume $q_0, q_1 < \infty$.

Consider

$$\frac{p_0}{q_0} \frac{q - q_0}{p - p_0} = \frac{p_0}{q_0} \frac{q_0}{p_0} \frac{\frac{q}{q_0} - 1}{\frac{p}{p_0} - 1} = \frac{q}{p} \cdot \frac{\frac{1}{q_0} - \frac{1}{q}}{\frac{1}{p_0} - \frac{1}{p}} = \frac{q}{p} \cdot \frac{\frac{1}{q_0} - (\frac{1-t}{q_0} + \frac{t}{q_1})}{\frac{1}{p_0} - (\frac{1-t}{p_0} + \frac{t}{p_1})} = \frac{q}{p} \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{p_0} - \frac{1}{p_1}}.$$

Also consider

$$\frac{p_1}{q_1} \frac{q - q_1}{p - p_1} = \frac{q}{p} \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{p_0} - \frac{1}{p_1}}.$$

Set

$$r = \frac{p_0}{q_0} \frac{q - q_0}{p - p_0} = \frac{p_1}{q_1} \frac{q - q_1}{p - p_1}.$$

We have

$$\|g_A|_{L^{p_0}}^{p_0} = p_0 \int_0^\infty \alpha^{p_0 - 1} \lambda_{g_A}(\alpha) \, d\alpha = p_0 \int_0^\infty \alpha^{p_0 - 1} \lambda_f(A + \alpha) \, d\alpha = p_0 \int_A^\infty (\beta - A)^{p_0 - 1} \lambda_f(\beta) \, d\beta$$

So

$$\|g_A\|_{p_0}^{p_0} \leq p_0 \int_A^\infty \beta^{p_0 - 1} \lambda_f(\beta) \, d\beta.$$

We have

$$\|h_A\|_{p_1}^{p_1} = p_1 \int_0^\infty \lambda_{h_A}(\alpha) \alpha^{p_1 - 1} \, d\alpha = p_1 \int_0^A \lambda_f(\alpha) \alpha^{p_1 - 1} \, d\alpha.$$

We also have

$$\|Tf\|_{q_0}^q = q \int_0^\infty \alpha^{q-1} \lambda_{T(f)}(\alpha) \, d\alpha = q \int_0^\infty (2\beta)^{q-1} \lambda_{Tf}(2\beta) \, d(2\beta).$$

Since $f = g_A + h_A$, we get that $|Tf| = |T(g_A + h_A)| \leq |Tg_A| + |Th_A|$, So

$$\lambda_{|Tf|}(2\beta) \leq \lambda_{Tg_A}(\beta) + \lambda_{Th_A}(\beta).$$

This lets us get

$$\|Tf\|_{q_0}^q \leq 2^{q-1} q \int_0^\infty \beta^{q-1} \left(\lambda_{Tg_A}(\beta) + \lambda_{Th_A}(\beta) \right) \, d\beta.$$

Use the weak-type condition with f replaced by g_A and with f replaced by h_A to conclude that

$$\|Tf\|_{L^q}^q \leq 2^{q-1} q \int_0^\infty \alpha^{q-1} \left(\left(\frac{c_0}{\alpha} \right)_0^q \|g_A\|_{p_0}^{q_0} + \left(\frac{c_1}{\alpha} \right)^{p_1} \|h_A\|_{p_1}^{p_1} \right) d\alpha$$

$$= 2^{q-1} q c_0^{q_0} \underbrace{\int_0^\infty \alpha^{q-1-q_0} \|g_A\|_{p_0}^{q_0} \, d\alpha}_{I} + 2^{q-1} q c_1^{q_1} \underbrace{\int_0^\infty \alpha^{q-1-q_1} \|h_A\|_{p_1}^{q_1} \, d\alpha}_{II} .$$

We have

$$I \leq \int_0^\infty \alpha^{q-1-q_0} \, d\alpha p_0^{q_0/p_0} \left(\int_A^\infty \beta^{p_0-1} \lambda_f(\beta) \, d\beta \right)^{q_0/p_0}$$

The above inequality holds for every $A > 0$. Let $r > 0$ and choose $A = \alpha^r$ (it will turn out that r is the value we computed earlier). We will finish the proof next time. □